On the Mysteries of Interpolation Jack Polynomials

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Thank you!

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Outline

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Partitions

Definition

A *partition* of a non-negative integer m is a tuple of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1 + \cdots + \lambda_n = m$. The number n is called the *length* of λ.

Concept: Many problems in representation theory lead to families of polynomials indexed by partitions. This is natural because in many situations representations are themselves indexed by partitions.

Interpolation Jack Polynomials

Interpolation Jack Polynomials

Interpolation Jack polynomials are certain symmetric polynomials P_λ, indexed by partitions λ, in n variables $x_1, \ldots, x_n$ and with coefficients in the field $\mathbb{Q}(k)$.

Example: when $n = 3$

The interpolation Jack polynomial associated with the partition $\lambda = (2, 0, 0)$ is

$$P_\lambda(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + \left(\frac{2k}{k+1}\right)(x_3x_2 + x_2x_1 + x_2x_1) - \left(\frac{6k^2+5k+1}{k+1}\right)(x_1 + x_2 + x_3) + \frac{9k^3+10k^2+3k}{k+1}$$

A Brief History

First defined by Knop and Sahi in 1996 as the unique symmetric polynomials satisfying

1. A degree condition
2. A vanishing condition
3. A normalization condition

A Brief History

Later in 1996, a combinatorial formula was found by Okounkov:

$$P_\lambda(x_1, \ldots, x_n) = \sum_{\substack{T \text{ a reverse} \\ \text{tableau} \\ \text{of shape } \lambda}} \psi_T(k) \prod_{s \in T} (x_{T(s)} + \varphi(s,k) - a'(s) + l'(s)k)$$

Why We Care

Given the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, we can construct its *universal enveloping algebra*, $\mathcal{U}$.

There is a bijective correspondence between the irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$ and those of $\mathcal{U}$.

The center of $\mathcal{U}$, $C(\mathcal{U})$, acts on the irreducible representations of $\mathcal{U}$ by scalars, and sometimes we can distinguish different representations based on these scalars.

There is a distinguished basis of $C(\mathcal{U})$ called the *Capelli elements*, b_λ, which is indexed by partitions.

The irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$ are also indexed by partitions, call them V_μ.

For $v \in V_\mu$ we have $b_\lambda \cdot v = P_\lambda^{k=1}(\mu) v$.

Why We Care

Works of Sahi, Salmasian and Serganova further show a connection between the derivative with respect to k of two-variable interpolation Jack polynomials and the eigenvalues of Capelli operators of orthosymplectic Lie superalgebras.

Our Goal:

We would like to find rational functions of k, c_λ^μ, such that

$$\frac{\partial}{\partial k} P_\lambda = \sum_\mu c_\lambda^\mu P_\mu$$

Note: These coefficients exist because the P_μ form a basis for the space of symmetric polynomials!

Example

For $\lambda = (2,0,0)$ we can write:

$$\frac{\partial}{\partial k} P_{(2,0,0)} = \left(\frac{5k+3}{k+1}\right) P_{(0,0,0)} - \left(\frac{6k+4}{k+1}\right) P_{(1,0,0)} + \left(\frac{2}{k^2+2k+1}\right) P_{(1,1,0)}$$

Inspiration

For P_λ with λ of length two, this problem was solved by Sahi, Salmasian and Serganova!

This work was necessary in obtaining formulas for eigenvalues of Capelli operators.

It also led to an interesting connection to a famous 100-year old hypergeometric identity, known as the Dougall-Ramanujan formula.

Outline

Summer Strategy

Over the summer of 2019 we looked at P_λ indexed by partitions of length three.

Strategy:

1. Use Sage to generate polynomials P_λ using the combinatorial formula.
2. Use Sage to iteratively find the coefficients c_λ^μ.
3. Try to find patterns in the generated coefficients.

Summer Results

We found several possible formulae!

Conjecture

If $\lambda = (D_1, 0, 0)$ and $\mu = (\mu_1, 0, 0)$ or $\mu = (0, 0, 0)$, then

$$c_\lambda^\mu = \frac{(-1)^{D_1 - \mu_1} \cdot \frac{D_1!}{(D_1 - \mu_1)\ \mu_1!} \times [(2k + \mu_1)^{\overline{D_1 - \mu_1}} + (k + \mu_1)^{\overline{D_1 - \mu_1}}]}{(k + \mu_1)^{\overline{D_1 - \mu_1}}}$$

Outline

First idea: Try to prove (some of) our summer conjectures!

However, I eventually went in a slightly different direction...

Monomial Symmetric Functions

Definition

The *monomial symmetric function* indexed by the partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is

$$m_\lambda(x_1, x_2, x_3) = \sum x_1^{a_1} x_2^{a_2} x_3^{a_3}$$

where the summation is over all distinct permutations $a = (a_1, a_2, a_3)$ of λ.

Example

The monomial symmetric function associated to $\lambda = (3, 2, 0)$ is

$$m_\lambda(x_1, x_2, x_3) = x_1^3 x_2^2 + x_1^3 x_3^2 + x_1^2 x_2^3 + x_1^2 x_3^3 + x_2^3 x_3^2 + x_2^2 x_3^3$$

Monomial Symmetric Functions

Fact: The monomial symmetric functions also form a basis for the space of symmetric polynomials.

A Few Definitions

Definition

The *size* of the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

Example

The size of $\lambda = (3, 2, 2)$ is $3 + 2 + 2 = 7$.

A Few Definitions

Definition

For partitions $\lambda = (\lambda_1, \ldots, \lambda_r), \mu = (\mu_1, \ldots, \mu_r)$ of the same length we write $\mu < \lambda$ in *lexicographic ordering* if for some index s

$$\lambda_j = \mu_j \text{ for } j < s \text{ and } \mu_s < \lambda_s$$

Example

If $\lambda = (3, 2, 1)$ and $\mu = (3, 2, 0)$ then $\mu < \lambda$.

A Few Definitions

Notation

Define $b_\lambda^\mu \in \mathbb{Q}(k)$ so that

$$P_\lambda = \sum b_\lambda^\mu m_\mu$$

Example

Example: Recall for $\lambda = (2,0,0)$ we have

$$\begin{aligned} P_\lambda(x_1,x_2,x_3) &= x_1^2 + x_2^2 + x_3^2 + \left(\tfrac{2k}{k+1}\right)(x_3x_2 + x_2x_1 + x_2x_1) \\ &\quad - \left(\tfrac{6k^2+5k+1}{k+1}\right)(x_1 + x_2 + x_3) \\ &\quad + \tfrac{9k^3+10k^2+3k}{k+1} \\ &= m_{(2,0,0)} + \left(\tfrac{2k}{k+1}\right) m_{(1,1,0)} - \left(\tfrac{6k^2+5k+1}{k+1}\right) m_{(1,0,0)} \\ &\quad + \left(\tfrac{9k^3+10k^2+3k}{k+1}\right) m_{(0,0,0)} \end{aligned}$$

and so

$$b_{(2,0,0)}^{(1,1,0)} = \frac{2k}{k+1}$$

A Theorem!

Theorem 1

For $\mu \geq \lambda$ we have $c_\lambda^\mu = 0$ and for $\mu < \lambda$ we have

$$c_\lambda^\mu = \frac{d}{dk} b_\lambda^\mu - \sum_{\substack{\mu < \nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu b_\nu^\mu$$

Conceptually: We can write c_λ^μ in terms of coefficients of monomial symmetric functions and the c_λ^ν with ν "between" μ and λ.

Proof of Theorem 1

Lemma 1

If $\nu \geq \lambda$ or $|\nu| > |\lambda|$ then $c_\lambda^\nu = 0$.

Proof idea: For $\nu \geq \lambda$ look at the leading term in the combinatorial formula, and recall that we can construct the coefficients c_λ^ν iteratively.
For $|\nu| > |\lambda|$ relate to Jack polynomials and recall that we can construct the c_λ^ν iteratively. ■

Proof of Theorem 1

Hence we can write

$$\frac{\partial}{\partial k} P_\lambda = \sum_{\substack{\nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu P_\nu$$

Equating the coefficient of m_μ on both sides, we see that

$$\frac{d}{dk} b_\lambda^\mu = \sum_{\substack{\nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu b_\nu^\mu$$

Proof of Theorem 1

Lemma 2

If $\nu < \mu$ then $b_\nu^\mu = 0$.

Proof idea: Look at the combinatorial formula. ■

Hence we can write

$$\frac{d}{dk} b_\lambda^\mu = \sum_{\substack{\mu \leq \nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu b_\nu^\mu$$

Proof of Theorem 1

Lemma 3

For any partition μ we have $b_\mu^\mu = 1$.

Proof idea: Relate to Jack polynomials. ∎

Hence we can write

$$\frac{d}{dk} b_\lambda^\mu = c_\lambda^\mu + \sum_{\substack{\mu < \nu < \lambda \\ |\nu| \leq |\lambda|}} c_\lambda^\nu b_\nu^\mu$$

Rearranging, we're done. ∎

Recall

Theorem 1

We can write

$$c_\lambda^\mu = \frac{d}{dk} b_\lambda^\mu - \sum_{\substack{\mu<\nu<\lambda \\ |\nu|\leq|\lambda|}} c_\lambda^\nu b_\nu^\mu$$

How do we use Theorem 1?

Technique to find the coefficient, c_λ^μ, of P_μ in $\frac{\partial}{\partial k} P_\lambda$:

1. Find the set of partitions ν such that $\mu < \nu < \lambda$ and $|\nu| \leq |\lambda|$.
2. For each of these ν, find c_λ^ν and b_ν^μ.
3. Find $\frac{d}{dk} b_\lambda^\mu$.
4. Use the coefficients in steps 1 and 2 along with theorem 1 to calculate c_λ^μ.

How do we use Theorem 1?

In general this can be quite difficult. However...

Theorem 2

For D that make sense we have,

$$
\begin{aligned}
c_{(D,0,0)}^{(D-1,0,0)} &= \frac{-3Dk - 2D(D-1)}{k+D-1} \\
c_{(D,0,0)}^{(D-1,1,0)} &= \frac{D(D-1)}{(k+D-1)^2} \\
c_{(D,0,0)}^{(D-2,2,0)} &= \frac{D(D-1)(D-2)(D-3)}{2(k+D-3)(k+D-2)^2(k+D-1)}
\end{aligned}
$$

A Further Conjecture

Conjecture

For all a, D that make sense we have,

$$c_{(D,0,0)}^{(D-a,a,0)} = \frac{D^{\underline{2a}}}{a(k+D-1)^{\underline{a}}(k+D-a)^{\underline{a}}}$$

This conjecture has been checked in Sage up to $D = 15$.

Another Theorem!

Theorem 3

The coefficient of $m_{(D-a,a,0)}$ in $P_{(D-b,b,0)}$ for all a, b, D that make sense is

$$b_{(D-b,b,0)}^{(D-a,a,0)} = \frac{(D-2b)^{\underline{a-b}}(k+a-b-1)^{\underline{a-b}}}{(a-b)!(k+D-2b-1)^{\underline{a-b}}}$$

Proof idea: Look at combinatorial formula.

Future Work

There is plenty of work still to be done! Some possibilities:

1. Continue as in my thesis.
2. Try to form a conjecture and then prove it using other methods.
3. Prove our summer conjectures.
4. Find the b_{λ}^{μ} (this won't solve the original problem but would be helpful).

Thank you!