Weak Normality

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July 25, 2021

Plane curves

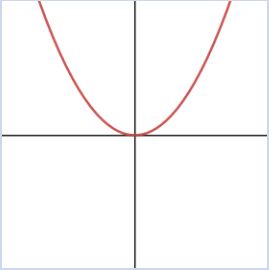
Let k be a field, and consider the vector space k^2 with coordinates x, y.

Definition

Let $f \in k[x, y]$ be a polynomial in the variables x, y. Then the set $\{(x, y) \in k^2 \mid f(x, y) = 0\} \subseteq k^2$ is called an *(affine) plane curve* over k.

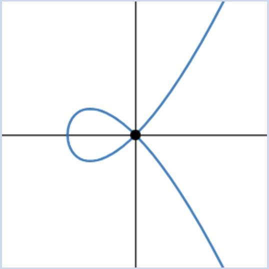
Plane curves

Examples



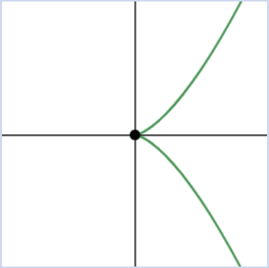
$f = y - x^2$

Parabola



$f = y^2 - x^3 - x^2$

Nodal cubic



$f = y^2 - x^3$

Cuspidal cubic

Plane curves

- A curve is *smooth* if the partial derivatives f_x, f_y don't simultaneously vanish anywhere.
  - The parabola is smooth since $f_y = 1$.
  - The nodal cubic and cuspidal cubic are both *singular* at $(0, 0)$.
- Smoothness (*normality*) doesn't distinguish between singularities, but *weak normality* does.
- Given a plane curve C over k, we can associate to it a k-algebra $k[C]$ called its *coordinate ring*. Then we say that C is weakly normal if $k[C]$ is weakly normal.

C	Parabola	Nodal cubic	Cuspidal cubic
f	$y - x^2$	$y^2 - x^3 - x^2$	$y^2 - x^3$
$k[C]$	$k[x, y]/(y - x^2)$	$k[x, y]/(y^2 - x^3 - x^2)$	$k[x, y]/(y^2 - x^3)$
Smooth/normal?	yes	no	no
Weakly normal?	yes	yes	no

Why weak normality?

Normality and weak normality are both notions of how 'nice' a curve is, but normal curves are 'nicer'. So why consider weak normality?

1. Normalization changes the underlying topological space of a curve, but weak normalization does not.

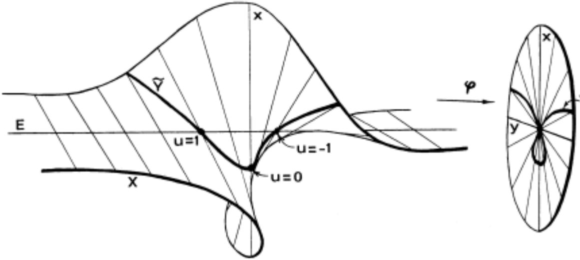


Figure: Normalization of a nodal curve

2. Cumino-Manaresi Theorem

Weakly normal varieties

Definitions

A **N-dimensional projective space** over an algebraically closed field k is defined as

$$\mathrm{P}_k^N := \frac{k^{N+1} - \{0\}}{k^*}$$

Figure: Image credit: Waldorf Projective Geometry collection

Weakly normal varieties

Definitions

- $X \subseteq \mathrm{P}_k^N$ is a **projective algebraic set** if it is defined as the vanishing locus of a set of homogeneous polynomials in the coordinates of P_k^N
- A **projective variety** is a projective algebraic set X such that we cannot write $X = X_1 \cup X_2$ for smaller projective algebraic sets X_1, X_2
- The **field of rational functions** on a projective variety X, denoted by $K(X)$, is the field of functions of the form $\frac{p(x_0,x_1,\ldots,x_N)}{q(x_0,x_1,\ldots,x_N)}$ where p and q are homogeneous polynomials in the co-ordinates of P_k^N of same degree, with $q \not\equiv 0$ on X.

Weakly normal varieties

Definitions (contd.)

- Projective varieties X and Y are **birational** if $K(X) = K(Y)$.
- A **hypersurface** is a projective variety defined by one polynomial.
- A projective variety X is **weakly normal** when, for every $U \subseteq X$ that is defined as the complement of a hypersurface, the subring of $K(X)$ consisting of elements defined everywhere on U is weakly normal.

Theorem (Cumino–Manaresi)

Every projective variety X over an algebraically closed field is birational to a weakly normal hypersurface $\tilde{X}$.

Consequence

Every curve X is birational to a curve with “good” nodes. A field-theoretic result implies X is birational to a hypersurface X'. By the assertion of weak normality, the singularities of the hypersurface X' cannot be too wild.

Rings

Definition (Ring)

A *ring* is a set R together with two binary operations $+, \cdot : R \times R \to R$ such that:

1. R is an abelian group under $+$;
2. For all $a, b, c \in R$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
3. There exists an element $1_R \in R$ such that for all $a \in R$ we have $1_R \cdot a = a$;
4. For all $a, b, c \in R$ we have

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and}$$
$$(a + b) \cdot c = a \cdot c + b \cdot c;$$

The ring R is *commutative* if for all $a, b \in R$ we have $a \cdot b = b \cdot a$.

Note: We will usually write simply ab for $a \cdot b$.

Example

Example

The integers $\mathbb{Z}$ form a ring under the usual addition and multiplication.

Hence forth, by "ring" we mean "commutative ring"!

Motivation for Total Quotient Ring

Elements in rings may not have multiplicative inverses!

Example

The element $2 \in \mathbb{Z}$ does not have a multiplicative inverse, because there is no $b \in \mathbb{Z}$ such that $2b = 1$.

Idea: For any ring R we want to find a ring $Q(R)$ such that every non-zero element of R has an inverse in $Q(R)$.

Integral Domains

Note: We will only define $Q(R)$ when R is an integral domain, but a similar definition works in the general case.

Definition

A ring R is an *integral domain* if $a, b \in R, ab = 0$ implies $a = 0$ or $b = 0$.

Example

The integers $\mathbb{Z}$ are an integral domain, but $\mathbb{Z}/n\mathbb{Z}$ is not an integral domain unless n is prime.

Total Quotient Ring

Definition

Let R be an integral domain, and let $S = R - \{0\}$. Then define $Q(R)$ to be the set $R \times S$ together with an equivalence relation. The equivelence class of (r, s) is denoted $\frac{r}{s}$ and we have:

$$\frac{r}{s} = \frac{r'}{s'} \iff (rs' - r's) = 0$$

We can make $Q(R)$ into a ring with the binary operations:

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$$
$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

Example

The ring $Q(\mathbb{Z})$ is the rational numbers $\mathbb{Q}$.

Total Quotient Ring

Key Idea: $Q(R)$ is the smallest ring such that:

- $R \subset Q(R)$; and
- Every non-zero element in R has an inverse in $Q(R)$.

For those familiar: $Q(R) = S^{-1}R$.

Reduced Rings

Definition (Reduced)

A ring R is *reduced* if $a \in R, a^n = 0$ implies $a = 0$.

Example

The integers $\mathbb{Z}$ are reduced. The rings $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$ are not reduced.

Ideals

Definition (Ideal)

Let R be a ring. A subset $I \subset R$ is an *ideal* if

- I is a subgroup of R under addition; and
- For all $r \in R, a \in I$ we have $ra \in I$.

Example

The set of integers consisting of multiples of 2 is an ideal of $\mathbb{Z}$.

Noetherian

Definition (Noetherian)

Let R be a ring. Then R is *Noetherian* if for every ascending chain of ideals

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

there exists an $n \in \mathbb{N}$ such that if $m > n$ then $I_m = I_n$.

Example

The following are Noetherian rings:

- Any field.
- The integers $\mathbb{Z}$.
- The ring $\mathbb{Z}/n\mathbb{Z}$ for any n.

Integral element

Definition (Integral element)

Let R be a ring, and let S be an R-algebra. An element $s \in S$ is integral over R if for some positive integer d we have that

$$s^d + r_1 s^{d-1} + \cdots + r_d \cdot 1_S = 0$$

for some suitable elements $r_j \in R$.

Example

Normalization

Definition (Normalization)

The normalization of the set of elements in S integral over R is a subring $R' \subseteq S$ of elements that are integral over R.

Example

The normalization of $\mathbb{R}$ in $\mathbb{C}$ is $\mathbb{C}$.

Seminormalization and Weak Normalization

Definition (Seminormalization and Weak Normalization)

Let R be a reduced Noetherian ring with total quotient ring $Q(R)$. The seminormalization of R in $Q(R)$ is the subring

$$^{+}R := \{x \in Q(R) : x^2, x^3 \in R\}$$

and the weak normalization is

$$^{*}R := \{x \in Q(R) : x^2, x^3 \in R \text{ or for some prime} p > 0 \text{ such that } x^p, px \in R\}$$

Observe that we have $^{+}R \subseteq^{*} R \subseteq R'$.

Normality, Seminormality, and Weak Normality

Definition (Normality, Seminormality, and Weak Normality)

Let R be a ring. We say that R is normal if $R = R'$, seminormal if $R =^{+} R$, and weakly normal $R =^{*} R$.

From the observation $R \subseteq^{+} R \subseteq^{*} R \subseteq R'$ above, we see that normality implies weak normality, which implies seminormality.

Example

The subring $k[x, y, x\sqrt{t}, y\sqrt{t}]$ inside $k(\sqrt{t})[x, y]$ is seminormal, but not weakly normal.

Maximal ideal

Definition (Maximal ideal)

A maximal ideal $\mathfrak{m}$ of a ring R is a proper ideal that is not contained in any other proper ideal.

Example

In $R = k[x, y]/(y^2 - x^2(x+1))$, which corresponds to the nodal curve, the element $t = \frac{y}{x}$ is not in R, but it is in the normalization of R through the equation $xt - y = 0$.

Local ring

Definition (Local ring)

A ring R is local if it contains a unique maximal ideal $\mathfrak{m}$. We will denote such a local ring by $(R, \mathfrak{m})$.

Example

Fields are local rings, since they contain a unique maximal ideal (0).
We can make any ring local at a multiplicative subset S containing 1 by formally inverting all elements in S.

Nonzerodivisors

Definition (Nonzerodivisors)

An element y in a ring R is called a nonzerodivisor if for any nonzero $a \in R$, we have $ay \neq 0$.

Example

Domains contain no nonzero nonzerodivisors.

Our Question

Our main question was inspired by a 2008 paper of Heitmann, which answered affirmatively the corresponding question for seminormality instead of weak normality.

Question

Let $(R, \mathfrak{m})$ be a reduced Noetherian local ring, and let $y \in \mathfrak{m}$ be a nonzerodivisor such that R/yR is weakly normal. Does it follow that R is weakly normal?

This question was also answered affirmatively in a 2020 paper of Murayama, extending a result of Bingener and Flenner, but this proof uses techniques from scheme theory which we wanted to avoid.

Proof Idea: Part 1

Proof.

We proceed by contradiction. Suppose $(R, \mathfrak{m})$ is a reduced Noetherian local ring, and there exists a nonzerodivisor $y \in \mathfrak{m}$ such that R/yR is weakly normal, but R is not weakly normal. If R is not weakly normal, there exists $x \in Q(R)$ such that $x \notin R$ and EITHER of the following conditions hold:

(I) $x^2, x^3 \in R$.

(II) There exists a prime number p such that $x^p, px \in R$.

□

Proof Idea: Part 2

Proof.

We then construct a ring B with $R \subset B \subset Q(R)$, and $x \in B$ with the following particularly nice properties:

(1) B is a finitely generated R module (A module is like a vector space, but over a ring instead of a field).

(2) The induced map $R/yR \to B/yB$ is an isomorphism.

□

Nakayama's Lemma

We need a lemma from commutative algebra in order to conclude the argument:

Lemma (Nakayama-Akizuki-Krull)

Suppose $(R, \mathfrak{m})$ is a local ring, $\mathfrak{a} \subset \mathfrak{m}$ is an ideal, and M is a finitely generated R module. If $\mathfrak{a}M = M$, then $M = 0$.

Proof Idea: Part 3

Proof.

We are almost ready to conclude the proof. Let $i : R \to B$ be the inclusion, and let $M = B/i(R)$ (One can check that M has has the structure of an R-module).
One can show that

$$M/yM \cong \frac{B/yB}{R/yR}.$$

However, one of our properties of B was that the map $R/yR \to B/yB$ was an isomorphism, so $M/yM = 0$. Therefore, $M = yM$. Since B is a finitely generated R-module, M is also a finitely generated R-module. By applying Nakayama's lemma with $\mathfrak{a} = yR$, we see that $M = 0$. This precisely means that i is an isomorphism, so $x \in R$, a contradiction. □