Joint Mathematics Meetings 2020

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On the Mysteries of Interpolation Jack Polynomials

Havi Ellers and Xiaomin Li

FIELDS CQAM

Background

Introduction

Interpolation Jack polynomials are certain polynomials P_λ in n variables $x_1, \dots, x_n$, indexed by partitions λ, and with coefficients in the field $\mathbb{Q}(\kappa)$ for a fourth variable κ. They were originally defined by Knop and Sahi in 1996[1], and later a combinatorial formula was developed by Okounkov[2]. In our research we primarily used the latter formula, given below:

$$P_\lambda(x_1, \dots, x_n) = \sum_{\substack{T \text{ a reverse} \\ \text{tableau} \\ \text{of shape } \lambda}} \psi_T(k) \prod_{s \in T} (x_{T(s)} - a'(s) + l'(s)\kappa)$$

Example. The interpolation Jack polynomial associated with the partition $\lambda = (2,0,0)$ is

$$P_{(2,0,0)}(x,y,z) = x^2 + y^2 + z^2 + \left(\frac{2\kappa}{\kappa+1}\right)(xy + xz + yz) - \left(\frac{6\kappa^2 + 5\kappa + 1}{\kappa+1}\right)(x+y+z) + \frac{9\kappa^3 + 10\kappa^2 + 3\kappa}{\kappa+1}$$

Relevance

Consider the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. We can form its *universal enveloping algebra*, denoted $\mathcal{U}(\mathfrak{gl}(n,\mathbb{C}))$, which is a quotient of the tensor algebra of $\mathfrak{gl}(n,\mathbb{C})$ by a particular two-sided ideal. In the 1990s, Okounkov defined a particular basis, s_λ, of the center of $\mathcal{U}(\mathfrak{gl}(n,\mathbb{C}))$ called the *quantum immanants*, which are indexed by partitions with at most n parts.

Since $\mathfrak{gl}(n,\mathbb{C})$ is also a group, we can consider its *representations*. A theorem of Cartan and Weyl states that the irreducible representations, V_μ, of $\mathfrak{gl}(n,\mathbb{C})$ are also indexed by partitions μ with at most n parts.

The action of $\mathfrak{gl}(n,\mathbb{C})$ on each V_μ then gives rise to an action of the s_λ on the V_μ. With this action, for any $v \in V_\mu$:

$$s_\lambda \cdot v = P_\lambda^{\kappa=1}(\mu) v$$

Thus interpolation Jack polynomials are interesting because they give the eigenvalues of elements of a basis for the center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{C})$ when that basis acts on irreducible $\mathfrak{gl}(n,\mathbb{C})$-modules.

More generally, the derivatives of interpolation Jack polynomials tend to appear when looking at actions on irreducible representations of Lie algebras.[4]

Goal

For all partitions $\lambda = (D_1, D_2, D_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$, we would like to find rational functions of κ, $c_\mu^\lambda(\kappa)$, such that

$$\frac{\partial}{\partial \kappa} P_\lambda(x,y,z) = \sum_\mu c_\mu^\lambda(\kappa) P_\mu(x,y,z)$$

Example 1. For $\lambda = (2,0,0)$ we can write:

$$\frac{\partial}{\partial \kappa} P_{(2,0,0)} = \left(\frac{5\kappa+3}{\kappa+1}\right) P_{(0,0,0)} + \left(\frac{-6\kappa-4}{\kappa+1}\right) P_{(1,0,0)} + \left(\frac{2}{\kappa^2 + 2\kappa + 1}\right) P_{(1,1,0)}$$

Results!

Which P_μ Appear

Conjecture 1. For partitions λ and μ, if

(a) $\sum \mu_i > \sum D_i$, or

(b) $\mu \geq \lambda$ in lexicographic ordering,

then P_μ appears with a zero coefficient in the linear combination for $\frac{\partial}{\partial \kappa} P_\lambda$. If λ has one or two non-zero parts then this is an if and only if condition.

In what follows we have $\mu_i, D_i > 0$ for all i, and we assume that the coefficients c_μ^λ are are not classified by conjecture 1. Furthermore, for $b > 0$ we define

$$a^{\overline{b}} := (a+0)(a+1)\cdots(a+b-1)$$

When $\lambda = (D_1, D_2, D_3)$:

Conjecture 2. If $\lambda = (D_1, D_2, D_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ then

$$c_\mu^\lambda = \begin{cases} 0 & \text{if } \mu_3 - D_3 < 0 \\ c^{(D_1 - D_3, D_2 - D_3, 0)}_{(\mu_1 - D_3, \mu_2 - D_3, \mu_3 - D_3)} & \text{otherwise} \end{cases}$$

When $\lambda = (D_1, 0, 0)$:

Conjecture 3. If $\lambda = (D_1, 0, 0)$ and $\mu = (\mu_1, 0, 0)$ or $\mu = (0,0,0)$, then

$$c_\mu^\lambda(\kappa) = \frac{(-1)^{D_1 - \mu_1} \cdot \frac{D_1!}{(D_1 - \mu_1)\, \mu_1!} \times \left[(2\kappa + \mu_1)^{\overline{D_1 - \mu_1}} + (\kappa + \mu_1)^{\overline{D_1 - \mu_1}}\right]}{(\kappa + \mu_1)^{\overline{D_1 - \mu_1}}}$$

Conjecture 4. If $\lambda = (D_1, 0, 0)$ and $\mu = (\mu_1, \mu_2, 0)$, then

$$c_\mu^\lambda(\kappa) = \frac{(-1)^{D_1 - \mu_1 + \mu_2} \cdot \frac{D_1!\, (D_1 - \mu_1 - 1)!}{(D_1 - \mu_1 - \mu_2)!\, (\mu_1 - \mu_2)!\, \mu_2!}}{(\kappa + \mu_1 - \mu_2 + 1)^{\overline{\mu_2}} \cdot (\kappa + D_1 - \mu_2)^{\overline{\mu_2}}}$$

Conjecture 5. If $\lambda = (D_1, 0, 0)$ and $\mu = (\mu_1, \mu_2, \mu_3)$, then

$$c_\mu^\lambda(\kappa) = \frac{(-1)^{D_1 - \mu_1 + \mu_2} \cdot \frac{D_1!(\mu_2 - 1)!}{(\mu_1 - \mu_2)!(\mu_2 - \mu_3)!(\mu_3)!} (\kappa - \mu_3 + 1)^{\overline{\mu_3}} \cdot (\kappa + \mu_1 - \mu_3 + 1)^{\overline{\mu_3 - 1}} \times S}{(\kappa + \mu_1 - \mu_2 + 1)^{\overline{D_1 - \mu_1 + \mu_2 - 1}} \cdot (\kappa + \mu_2 - \mu_3 + 1)^{\overline{\mu_3}} \cdot (2\kappa + \mu_1 - \mu_3 + 1)^{\overline{\mu_3}}}$$

where S =

$$\sum_{j=0}^{\substack{D_1 - \mu_1 \\ -\mu_2 - \mu_3}} \binom{D_1 - \mu_1 - \mu_2 - 1 - j}{\mu_3 - 1} \binom{j + \mu_2 - 1}{\mu_2 - 1} (2\kappa + \mu_1)^{\overline{D_1 - \mu_1 - \mu_2 - \mu_3 - j}} \cdot (\kappa + D_1 - \mu_2 - j)^{\overline{j}}$$

When $\lambda = (D_1, D_2, 0)$:

Conjecture 6. If $\lambda = (D_1, D_2, 0)$ and $\mu = (\mu_1, 0, 0)$ or $\mu = (0,0,0)$, then

If $D_1 - \mu > 0$:

$$c_\mu^\lambda(\kappa) = \frac{(-1)^{D_1 - \mu_1} \left[\frac{(D_1 - D_2)!}{\mu_1!}\right] [(D_2 - 1)!]\, [(\kappa - 1)(\kappa)]\, [(2\kappa + \mu_1)^{\overline{D_1 - \mu_1 - D_2}}] \times S}{(\kappa + \mu_1 - D_2)^{\overline{D_1 - \mu_1}} \cdot (2\kappa + D_1 - D_2)^{\overline{D_2 - (D_1 - \mu_1)}}}$$

where S has:

$$\text{leading term} = \binom{D_1 - \mu_1 - 1}{D_2 - 1} \kappa^{2(D_2 - 1)}$$

$$\text{constant term} = (-1)^{D_2 - 1} \cdot D_2 \cdot (\mu_1 - D_2 + 1)^{\overline{D_2 - 1}} \cdot (D_1 - D_2 + 1)^{\overline{D_2 - 1}}$$

If $D_1 - \mu_1 = 0$:

$$c_\mu^\lambda = (-1)^{D_2} \cdot (D_2 - 1)!$$

Conjecture 7. If $\lambda = (D_1, D_2, 0)$ and $\mu = (\mu_1, \mu_2, 0)$, then

If $\mu_2 > D_2$:

$$c_\mu^\lambda(\kappa) = \frac{(-1)^{D_1 + D_2 - \mu_1 + \mu_2} \cdot \frac{(D_1 - D_2)!\, (D_1 - \mu_1 - 1)!}{(\mu_1 - \mu_2)!\, (\mu_2 - D_2)!\, (D_1 + D_2 - \mu_1 - \mu_2)!}}{(\kappa + \mu_1 - \mu_2 + 1)^{\overline{\mu_2 - D_2}} \cdot (\kappa + D_1 - \mu_2)^{\overline{\mu_2 - D_2}}}$$

Note: To generate these conjectures, we wrote code in Sage to generate the polynomials and linear combinations, and applied various manual techniques to discover the formulas. (See [3].)

Future Work

- Complete the cases where we only have partial formulae.
- Try to prove our conjectures.
- Try to generalize to n variable case!

References:

[1] Friedrich Knop and Siddartha Sahi. Difference equations and symmetric polynomials defined by their zeros. *arXiv preprint q-alg/9610017*, 1996.

[2] Andrei Okounkov and Grigori Olshanski. Shifted Jack polynomials, binomial formula, and applications. *arXiv preprint q-alg/9608020*, 1996.

[3] Havi Ellers and Xiaomin Li. Lie algebras report. Available at https://mysite.science.uottawa.ca/hsalmasi/report/report-FUSRP2019.pdf

[4] Hadi Salmasian, pers. com.

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