Texas A&M University REU in Number Theory

Effective Bounds for Traces of Maass-Poincaré Series

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Background and Definitions

An important problem in Number Theory is that of bounding the Fourier coefficients of modular forms. Work of Duke and Jenkins, and Miller and Pixton, shows that the generating functions for traces of Maass-Poincaré series appear as holomorphic parts of certain half-integral weight weakly holomorphic modular forms for $\Gamma_0(4)$. Our goal was to find an effective upper bound for traces of Maass-Poincaré series, which can then be used to help bound the Fourier coefficients of the aforementioned half-integral weight modular forms.

The Complex Upper Half Plane

Definition 1. *The **special linear group of degree 2 over** $\mathbb{Z}$ is the group*

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \alpha,\beta,\gamma,\delta \in \mathbb{Z} \text{ and } \alpha\delta - \beta\gamma = 1 \right\}.$$

Definition 2. *The **complex upper half plane** is the set*

$$\mathbb{H} := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by the left group action

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Definition 3. *A function $f : \mathbb{H} \to \mathbb{C}$ is **$\mathrm{SL}_2(\mathbb{Z})$-invariant** if $f(Mz) = f(z)$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$.*

Binary Quadratic Forms

Definition 4. *A **binary quadratic form** is a homogeneous polynomial $Q : \mathbb{Z}^2 \to \mathbb{Z}$ such that*

$$Q(x,y) = a_Q x^2 + b_Q xy + c_Q y^2$$

for some $a_Q, b_Q, c_Q \in \mathbb{Z}$.

Definition 5. *The **discriminant** of a form Q is $d := b_Q^2 - 4a_Q c_Q$, and the set of all quadratic forms of discriminant d is denoted $\boldsymbol{Q_d}$.*

There is a right group action of $\mathrm{SL}_2(\mathbb{Z})$ on Q_d given by

$$Q \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x,y) = Q(\alpha x + \beta y, \gamma x + \delta y).$$

The quotient $Q_d/\mathrm{SL}_2(\mathbb{Z})$ is always finite, and the orbit of a form $Q \in Q_d$ under this action is denoted by $[Q]$.

Definition 6. *The **class number** of d is $h(d) := |Q_d/\mathrm{SL}_2(\mathbb{Z})|$.*

The Trace

Definition 7. *Given a form $Q \in Q_d$, the **CM point associated with** Q is*

$$\tau_Q := \frac{-b_Q + i\sqrt{|d|}}{2a_Q}.$$

We can choose a set of forms $Q_1, \ldots, Q_{h(d)}$ such that $[Q_1], \ldots, [Q_{h(d)}]$ are distinct equivalence classes, and such that for all $1 \le i \le h(d)$,

$$\mathrm{Im}(\tau_{Q_i}) \ge \frac{\sqrt{3}}{2}.$$

Definition 8. *Given a $\mathrm{SL}_2(\mathbb{Z})$-invariant function f, the **trace** of f is*

$$\mathrm{Tr}_d(f) := \sum_{i=1}^{h(d)} f(\tau_{Q_i}).$$

Note it can be shown that the trace is well-defined if f is $\mathrm{SL}_2(\mathbb{Z})$-invariant.

Poincaré Series

Definition 9. *For $\nu \in \mathbb{Z}^+$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the **Maass-Poincaré series***

$$F_{s,\nu}(z) := 2\pi\nu^{s-\frac{1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi\nu \mathrm{Im}(\gamma z))\, e(-\nu \mathrm{Re}(\gamma z))$$

where $I_{s-1/2}$ is the I-Bessel function of order $s - \frac{1}{2}$, $e(z) := e^{2\pi i z}$ and Γ_∞ is the subgroup of translations in $SL_2(\mathbb{Z})$.

Result

Theorem 1. *Let $s \ge 1$ be a real number and $\nu \in \mathbb{Z}^+$. Then*

$$|\mathrm{Tr}_d(F_{s,\nu})| \le C(s,\nu) h(d) e^{\pi\nu\sqrt{|d|}}.$$

where $C(s,\nu)$ is an explicit function of s and ν.

Corollary 1. *In particular, if j is the classical modular j-function given by the Fourier expansion*

$$j(z) := q^{-1} + 744 + 196884q + \cdots, \quad q := e(z),$$

and we define $J(z) := j(z) - 744$, then we have

$$|\mathrm{Tr}_d(J)| \le (1.73 \times 10^6) h(d) e^{\pi\sqrt{|d|}}.$$

Key Ideas

- The Maass-Poincaré series $F_{s,\nu}(z)$ is periodic and thus has a Fourier expansion, which is given by

$$F_{s,\nu}(z) = 2\pi\nu^{s-\frac{1}{2}} y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi\nu y) e(-\nu x) + \frac{4\pi^{1+s}\sigma_{2s-1}(\nu)}{(2s-1)\Gamma(s)\zeta(2s)} y^{1-s} + 4\pi\nu^{s-\frac{1}{2}} \sum_{n \ne 0} b(n,\nu;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) e(nx) \quad (1)$$

  where $x := \mathrm{Re}(z)$, $y := \mathrm{Im}(z)$,

$$b(n,\nu;s) := \sum_{c>0} \frac{S(n,-\nu;c)}{c} \begin{cases} I_{2s-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right) & n > 0 \\ J_{2s-1}\left(\frac{4\pi\sqrt{|n|\nu}}{c}\right) & n < 0, \end{cases}$$

$$S(a,b;c) := \sum_{\substack{d \bmod c \\ (c,d)=1}} \left(\frac{a\bar{d} + bd}{c}\right)$$

  is the ordinary Kloosterman sum, Γ is the gamma function, ζ is the Riemann zeta function, σ_{2s-1} is a divisor function, and I_r, J_r, and K_r are the I, J, and K Bessel functions, respectively, of order r.
- We then bound the absolute value of each term in (1). The main techniques used are considering asymptotic expressions for the I, J, and K Bessel functions, the Weil Bound, i.e.

$$|S(n,-\nu;c)| \le \tau(c)(n,-\nu,c)^{1/2} c^{1/2}$$

  where τ is the divisor function, and the fact that $\mathrm{Im}(\tau_{Q_i}) \ge \frac{\sqrt{3}}{2}$ for all $1 \le i \le h(d)$. This results in the bound

$$|F_{s,\nu}(\tau_{Q_i})| \le C(s,\nu) e^{\pi\nu\sqrt{|d|}}$$

- Inserting (1) into the expression for $Tr_d(F_{s,\nu})$, we then obtain

$$|Tr_d(F_{s,\nu})| \le \sum_{i=1}^{h(d)} C(s,\nu) e^{\pi\nu\sqrt{|d|}}.$$

  Observing that the summand no longer depends on the index i, we obtain the desired bounds.
- The corollary follows from the identity: $J(z) = F_{1,1}(z) - 24$.

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References: 1. Duke, W., Jenkins, P. *Integral traces of singular values of weak Maass forms*. Algebra Number Theory 2 (2008) 573-593
2. Folsom A., Masri R.: *The asymptotic distribution of traces of Maass-Poincaré series*. Adv. Math. **226**, 3724-3759 (2011)
3. Miller, A., Pixton, A.: *Arithmetic traces of non-holomorphic modular invariants*. Int. J. Number Theory **6**(1), 69-87 (2010)
4. Niebur, D. *A class of nonanalytic automorphic functions*, Nagoya Math. J. 52 (1973) 133-145.